INTRODUCTION TO ARITHMETIC DYNAMICS

CHARLES FAVRE

7 times one hour twenty minutes

1. Introduction: What is arithmetic dynamics?

(15 minutes)

- foreword: unusual course (some proofs are omitted, some introduction to a wide subject)

- notion of (pre)-periodic points, of prime period; ω-limit set

2. Rational function in one variable (over any field)

(1h05)

- $f(T) = \frac{P(T)}{Q(T)}$, $\mathbb{P}^1(K) = K \cup \{\infty\}$, poles and zeroes

- number of preimages, number of critical points (K of char. 0)

- estimates on the number of (pre)-periodic points

- when K is alg. closed, the number of periodic points in infinite

End of talk 1

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3. Preperiodic points over a number field K

(30 minutes)

- Theorem 1: $f \in K(T)$ admits finitely many preperiodic points over K

- discussion when K is alg. closed,

- examples: monomial maps

- example: quadratic maps over $K = \mathbb{R}$

- a concrete example : $P(z) = z^2 - \frac{29}{16}$ has 8 preperiodic points, and one orbit of period 3

- discuss Poonen's conjecture for quadratic polynomials with $c \in \mathbb{Q}$: no $\mathbb{Q}$-periodic points of period ≥ 4, and the number of preperiodic points in $\mathbb{Q}$ is ≤ 8 (excluding infinity).

Date: November 22, 2018.

4. The uniform boundedness conjecture

(40 minutes)

- The uniform boundedness conjecture (statement)

- some partial results (Benedetto)

- Weierstrass model for elliptic curves

- Descent of the doubling map to $\mathbb{P}^1$, preperiodic points

- Mordell-Weil (statement)

- UBC and Merel's theorem

End of talk 2

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5. Archimedean and non-Archimedean norms

(70 minutes)

- valued/metrized/normed fields

- Archimedean examples : $\mathbb{R}$, $\mathbb{C}$,

- classification of complete Archimedean fields

- basic non-Archimedean fields : definition, $\mathbb{Q}_p$, residue field

- p-adic fields: norm of finite extensions of $\mathbb{Q}_p$,

- construction of $\mathbb{C}_p$, and proof it is algebraically closed

6. Norms on number fields

- Ostrowski's theorem and places in $\mathbb{Q}$ (10 minutes)

End of talk 3

- norms on number fields: the Archimedean case. Local product formula. Proof of the classification.

- norms on number fields: the non-Archimedean case. Local product formula. Proof of the classification.

- the global product formula

7. Heights

(50 minutes)

- naive definition in terms of minimal polynomial

- interpretation over $\mathbb{Q}$ in terms of absolute values

- definition of h in terms of norms on K, and in terms of the Galois conjugates of an element

End of talk 4

- Northcott property

8. Dynamical heights

(1h00)

- Kronecker theorem

- dynamical height for a rational map in one variable

- proof of Theorem 1.

- refined conjecture using heights: dynamical Lehmer's conjecture 3.25

9. Complex Fatou/Julia theory for polynomials

(20 minutes)

$P \in \mathbb{C}[T]$

- g_P, $J(P)$, $K(P)$

- $K(P)$ connected iff all critical points have bounded orbits

- periodic points and $J(P)$

- comments on the dynamics in the Fatou set

End of talk 5

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10. The Berkovich affine line

(1h20)

- Berkovich line using multiplicative semi-norms

- type of points

- link with balls

End of talk 6

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11. THE NON-ARCHIMEDEAN FATOU/JULIA THEORY FOR POLYNOMIALS

(40 minutes)

- action of P on the Berkovich affine line; $P(x_{\bar{B}}) = x_{P(\bar{B})}$.

- g_P, $K(P)$, $J(P)$

- $K(P)$ connected iff P has potential good reduction

12. BENEDETTO'S RESULT ON UBC

(40 minutes)

- Statement: N, d, s fixed. There exists a constant $C > 0$ such that any polynomial of degree $\leq d$ defined over a number field of degree $\leq N$ and having $\leq s$ bad places has at most $Cs \log s$ preperiodic points over K.

- idea of proof for $d = 2$, $P_c(z) = z^2 + c$.

End of talk 7

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13. REFERENCES

(1) J. Silverman. The arithmetic of dynamical systems. Chapter 3.
(2) A. Robert. A course in p-adic analysis. Chapters 1, 2 & 3.
(3) Koblitz. p-adic numbers, p-adic analysis and ζ-functions. Chapter I & III.
(4) M. Baker and R. Rumely.

14. LIST OF EXERCICES AND PROBLEMS

Rational functions.

(A.1) Suppose $f, g \in K(T)$ are two rational functions of positive degree. Prove that $\deg(f \circ g) = \deg(f) \times \deg(g)$. [ASSIGNMENT]

(A.2) Suppose K is a field of characteristic 0, and let $f \in K[T]$ be a polynomial of degree $d \geq 1$. Prove that
$$\sum_{x \in \mathbb{P}^1(K)} (\deg_x(f) - 1) = 2d - 2 \ .$$
Indication: $\deg_\infty(f) = d$ and for any $x \in K$, we have $\operatorname{ord}_x(f') = \deg_x(f) - 1$.

(A.3) Let K be a field of characteristic $p > 0$. Prove that $\deg_x(F) = p$ for any $x \in \mathbb{P}^1(K)$ when $F(T) = T^p$ is the Frobenius map.

(A.4) Suppose K is an algebraically closed field of characteristic $p > 0$, and $f \in K(T)$ has degree $d \geq 2$. Show that the set of periodic points of period prime to p is infinite.

Periodic points.

(B.1) Prove that z^2+c has a finite number of real pre-periodic points when $c>3/4$.

(B.2) Prove that z^2-2 is conjugated to the Tchebyshev polynomial $T_2(z+\frac{1}{z})=z^2+\frac{1}{z^2}$; and show that z^2-2 has infinitely many real periodic points.

(B.3) Prove that z^2+c has infinitely many pre-periodic points when $c<-2$. (Indication: construct two disjoint closed segments I_+ and I_- such that $P(I_\pm)\subset I_+\cup I_-$ and use symbolic dynamics).

(B.4) Prove that $z^2-\frac{13}{9}$ has exactly 6 pre-periodic points in $\mathbb{Q}$. [ASSIGNMENT]

p-adic fields.

(C.1) Let $K/\mathbb{Q}_p$ be any finite extension. Write $f_K=[\tilde{K}:\mathbb{F}_p]$, and introduce $e_K\in\mathbb{N}^*$ such that $|K^*|=p^{\mathbb{Z}/e_K}$. Show that $e_Kf_K\leq[K:\mathbb{Q}_p]$. [ASSIGNMENT]

(C.2) Let $K/\mathbb{Q}_p$ be any finite extension, and choose a basis $K=\mathbb{Q}_pe_1+\ldots+\mathbb{Q}_pe_n$. Put the product norm on K. Show that the function $f(x)=|N_{K/\mathbb{Q}_p}(x)|$ is continuous on K. [ASSIGNMENT]

(C.3) Show that $\mathbb{Q}_p^{\mathrm{alg}}$ is not complete. To that end introduce $F_n=\{x\in\mathbb{Q}_p^{\mathrm{alg}},\deg(x)\leq n\}$. Show that it is a closed set which has empty interior. Conclude using Baire theorem.

(C.4) Let K be any non-Archimedean normed field, and let K^{alg} be an algebraic closure of K. Recall from the lectures that K^{alg} is endowed with a unique norm extending the one on K. Show that the residue field of K^{alg} is an algebraic closure of the residue field of K.

(C.5) Let K be any non-Archimedean normed field. Show that K is locally compact iff its residue field should be finite and its value group $|K^*|$ is a discrete subgroup of $(\mathbb{R}_+^*,\times)$.

(C.6) Show that a non-trivial Archimedean norm on $\mathbb{Q}$ is equal to $|\cdot|^\epsilon$ for some $\epsilon\in(0,1]$ where $|\cdot|$ is the standard norm.

Norms on number fields.

(D.1) Suppose K is a finite extension of $\mathbb{Q}$ and let $|\cdot|$ be any norm on K whose restriction to $\mathbb{Q}$ is the p-adic norm for some prime number p. Let $\hat{K}$ be the completion of $(K,|\cdot|)$. Prove that the extension $\hat{K}/\mathbb{Q}_p$ is finite. (indication: write $K=x_1\mathbb{Q}\oplus\ldots\oplus x_n\mathbb{Q}$ and prove that $x_1\mathbb{Q}_p\oplus\ldots\oplus x_n\mathbb{Q}_p\subset\hat{K}$ is complete) [ASSIGNMENT]

(D.2) Consider the degree 2 extension $K=\mathbb{Q}[i]$ over $\mathbb{Q}$. Pick any prime number p. Show that the set of places on K extending $|\cdot|_p$ has either one or two elements depending on whether T^2+1 splits in $\mathbb{Q}_p$ or not. (the latter appears iff p is congruent to 1 modulo 4).

Heights.

(E.1) Give an explicit bound in terms of d and N on the number of points $x\in\mathbb{Q}^{\mathrm{alg}}$ such that $\deg(x)\leq d$ and $h(x)\leq N$.

(E.2) Write $c(N)=\mathrm{Card}\{x\in\mathbb{Q},h(x)\leq N\}$. Find constant b_1,b_2 such that $b_1N^2\leq c(N)\leq b_2N^2$, and show that $c(N)/N^2\to\pi^2/6$ as $N\to\infty$.

(E.3) Recall that the modified height of $x\in\mathbb{Q}^{\mathrm{alg}}$ is defined as

$$\bar{h}(x):=\frac{1}{\deg(x)}\log\max\{|a_0|,\ldots,|a_d|\}$$

where $a_0T^d+\ldots+a_d$ is the minimal polynomial of x with $a_i\in\mathbb{Z}$ and $\gcd\{a_i\}=1$. Prove that $|h(x)-\bar{h}(x)|$ is bounded by a universal constant (independent on d). [ASSIGNMENT]

(E.4) Compute the height of $\sqrt{2}$ and $2+i$.

The Berkovich affine line. In this section, we fix any non-Archimedean algebraically closed complete metrized field $(K, |\cdot|)$,

(F.1) Pick any polynomial $P \in K[T]$. Show that there exists a finite subset $\mathcal{E} \subset \tilde{K}$ such that $\sup_{\bar{B}(0,1)} |P| = |P(z)|$ for all $z \in K^0$ whose reduction does not belong to $\mathcal{E}$.

(F.2) Pick any polynomial $P = a_1 T + \ldots + a_d T^d \in K[T]$ and any real number $r \geq 0$. Show that $P(\bar{B}(0,r)) = \bar{B}(0, \max\{|a_i| r^i\})$. Indication: treat first the case $r = 1$, pick $|w| \leq \max\{|a_i| r^i\}$ and look at the d solutions of the equation $P(z) = w$. [ASSIGNMENT]

Dynamics on non-Archimedean fields. In this section, we fix any non-Archimedean complete metrized field $(K, |\cdot|)$,

(G.1) Suppose the residual characteristic of K is 2, and consider a quadratic polynomial $P_c(z) = z^2 + c$ with $1 < |c| \leq 4$.
- Show that the critical point escapes to infinity.
- Compute the norm of the two fixed points $x_\pm$ of P_c.
- Show that the ball B centered at x_+ of radius $|x_+ - x_-|$ which is totally invariant.
- Deduce that P_c has potential good reduction.

(G.2) Suppose the residual characteristic of K is p, and pick any polynomial $P \in K[T]$ of degree d. We assume that P has good reduction.
- Show that all fixed points of P are attracting (the derivative at the fixed point has norm < 1) when p divides d.
- Show that for any $\zeta \in K^0$ fixed by $\tilde{P}$ there exists at least one fixed point z of P whose reduction is equal to ζ, and show that this fixed point is unique when d is prime to p.

(G.3) Show that the Julia set of any polynomial is totally discontinuous by proving that any two points in $J(P)$ are not comparable. [ASSIGNMENT]

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